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LARGE DEFLECTION AXISYMMETRIC ANALYSIS OF ORTHOTROPIC ANNULAR PLATES ON ELASTIC FOUNDATIONS

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Abstract—Large deflection axisymmetric response analyses of cylindrically orthotropic thin annular plates, resting on annular elastic foundations and subjected to uniformly distributed loads are presented. Static and step function loads applied to clamped and simply supported annular plates are considered. The natural boundary conditions employed at the hole are consistent with those obtained from Hamilton's principle. Von Kármán type governing equations are solved using the orthogonal point collocation method in conjunction with the Newmark-β scheme. The maximum response to step loads obtained from static analysis is shown to agree very well with that obtained from a transient analysis. Thus one static analysis is sufficient to obtain the maximum response to a whole set of step loads, instead of separate transient solutions for each load. New results are presented to study the effects of the foundation parameters on the plate response.

1. INTRODUCTION

Annular plate structures on elastic foundations are encountered in foundations of large storage tanks, deep sea pressure vessels and heavy duty machines. Large dynamic loads necessitate large deflection dynamic analysis. The large deflection static response of an isotropic circular plate on Winkler foundation has been studied by Sinha (1963) and Datta (1974) using Berger's approximation. Bolton (1972) has employed the Galerkin technique with several terms for the large deflection static analysis of an isotropic plate on a Winkler foundation. The large amplitude free vibration of plates on a Winkler foundation has been analysed by Gajendra (1967) employing Berger's approximation and the Galerkin technique. The large deflection transient response of an isotropic circular plate on a Pasternak foundation for the case of step loads has been presented by Nath (1982) using the Chebyshev series and the Houbolt technique.

The large deflection transient response of an orthotropic annular plate on a Winkler–Pasternak foundation has been studied by Nath and Jain (1983) using the Chebysev series and the Houbolt technique. Immovable clamped and simply supported annular plates with a free edge at the hole have been analysed for one value of the step load. The free edge condition used by Nath and Jain (1983) corresponds to the condition of a zero shear force in the plate and it is valid if the Pasternak foundation beneath the plate is continuous.

The objective of this study is to obtain the large deflection axisymmetric response of cylindrically orthotropic thin annular plates resting on annular elastic foundations using the consistent natural boundary condition of zero generalized shear force (Vlasov and Leontiev, 1966) at the free inner edge of the plate and the foundation. Immovable clamped and simply supported plates subjected to uniformly distributed static and step function loads are considered. Based on an energy balance, the maximum dynamic response to step loads is obtained from the results for static loads. One static analysis is sufficient to get the maximum dynamic response to a whole set of step loads instead of separate transient solutions for each load. In order to establish the accuracy of this method, transient analyses have also been conducted for typical loads for comparison. A three parameter non-linear model of the foundation is employed. The foundations are assumed to be capable of exerting pull and push, i.e. the plate is attached to the foundation at every point. Results are presented for linear Winkler, Pasternak and non-linear Winkler foundations and the effects of geometric nonlinearity and the foundation parameters are investigated.

2. GOVERNING EQUATIONS

In the Winkler model it is assumed that the foundation applies to the plate distributed reaction p normal to the plate, which is proportional to the plate deflection W, so that

$$p = -kW \tag{1}$$

where k is the Winkler parameter. Pasternak (1954) included the effect of shear deformation of the foundation. For the case of axisymmetric deflection of a circular plate, the reaction p for the Pasternak foundation is given by

$$p = -kW + \frac{1}{r}(grW_{,r})_{,r} \tag{2}$$

where k and g are the two foundation parameters. The reaction for a non-linear Winkler foundation, which is modelled as in Massalas and Kafousias (1979), in terms of parameters k and k_1, is given by

$$p = -kW - k_1 W^3. \tag{3}$$

In this paper, the foundation has been modelled in terms of three parameters k, k_1, g in order to study plates on linear Winkler, Pasternak and non-linear Winkler foundations. The foundation reaction is taken as

$$p = -kW - k_1 W^3 + \frac{1}{r}(grW_{,r})_{,r}. \tag{4}$$

The inertia of the foundation is accounted for in the sense of Vlasov and Leontiev (1966) by using an effective mass density for the plate. The transverse displacement $W_f(r, z, t)$ in the single layer foundation of depth H is assumed as

$$W_f(r, z, t) = \psi(z) W(r, t). \tag{5}$$

The kinetic energy of an area element dA of the plate and the foundation can be expressed as

$$\tfrac{1}{2} h\gamma \dot{W}^2 \, \mathrm{d}A + \tfrac{1}{2}\int_0^H \gamma_0 \psi^2(z) \, \mathrm{d}z \, \dot{W}^2 \, \mathrm{d}A = \tfrac{1}{2} h\gamma^* \dot{W}^2 \, \mathrm{d}A \tag{6}$$

where h is the thickness of the plate and γ, γ_0 are the mass densities of the plate and the foundation, respectively. Thus the effective mass density γ^* of the plate is given by

$$\gamma^* = \gamma + \int_0^H \gamma_0 \psi^2(z) \, \mathrm{d}z / h. \tag{7}$$

The generalized shear force V_r is related to the stress resultants M_r, M_θ, N_r, N_θ and the foundation modulus g by

$$V_r = M_{r,r} + \frac{1}{r}(M_r - M_\theta) + (N_r + g) W_{,r}. \tag{8}$$

The last term in this expression is the contribution of the in-plane force in the plate and the shear in the foundation.

The equations of motion and compatibility for moderately large axisymmetric deflections of a cylindrically orthotropic thin circular plate are given by Chia (1980) in terms of

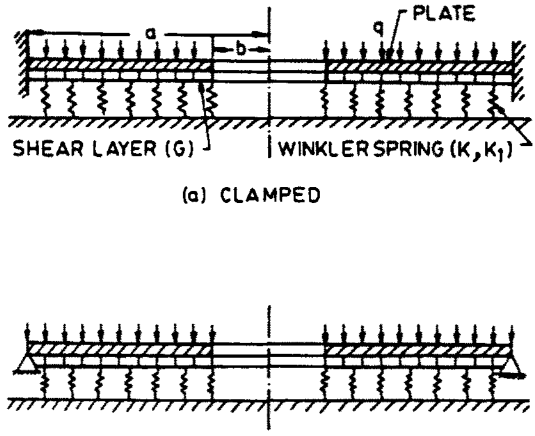

Fig. 1. Geometry of annular plates on elastic foundation.

the deflection W and the stress function Φ. Including the reaction and the inertia of the foundation, these equations may be expressed as

$$\Phi_{,rr}+\frac{1}{r}\Phi_{,r}-\frac{\beta}{r^2}\Phi+\frac{hE_\theta}{2r}(W_{,r})^2=0$$

$$D\left[rW_{,rrr}+W_{,rr}-\frac{\beta}{r}W_{,r}\right]-W_{,r}(\Phi+gr)=\int_b^r(q-kW-k_1W^3-\gamma^*hW_{,tt})r\,\mathrm{d}r-\frac{V_r(b)}{2\pi}$$

where

$$N_r=\frac{\Phi}{r},\quad N_\theta=\Phi_{,r},\quad \beta=\frac{E_\theta}{E_r}=\frac{\nu_\theta}{\nu_r} \tag{9}$$

$$D=\frac{E_\theta h^3}{12(\beta-\nu_\theta^2)} \tag{10}$$

and a, b are the outer and inner radii of the plate (Fig. 1). The elastic moduli and Poisson's ratios are E_r, E_θ and ν_r, ν_θ, and β is the orthotropic parameter. The uniformly distributed load is q. The generalized shear force for an annular plate attached to an elastic foundation at a free hole is

$$V_r(b)=-D\left(W_{,rrr}+\frac{1}{r}W_{,rr}-\frac{\beta}{r^2}W_{,r}\right)+\left(\frac{\Phi}{r}+g\right)W_{,r}=0. \tag{11}$$

Introducing the dimensionless variables

$$w=\frac{W}{h},\quad \phi=\frac{(a-b)}{D}\Phi,\quad \xi=\frac{b}{(a-b)},\quad \rho=\frac{r-b}{a-b},\quad \tau=\left[\frac{D}{\gamma^*ha^4}\right]^{1/2}t,$$

$$Q=\frac{qa^4}{E_rh^4},\quad K=\frac{Ka^4}{E_rh^3},\quad K_1=\frac{k_1a^4}{E_rh},\quad G=\frac{ga^2}{E_rh^3} \tag{12}$$

and using equation (11) reduces the governing equations (9) to the following dimensionless form:

$$(\rho+\xi)^2\phi''+(\rho+\xi)\phi'-\beta\phi+6(\beta-v_\theta^2)(\rho+\xi)w'^2=0$$

$$(\rho+\xi)^2 w'''+(\rho+\xi)w''-\beta w'-(\rho+\xi)w'\phi+\frac{12(\beta-v_\theta^2)}{\beta(1+\xi)^4}(\rho+\xi)\left[\int_0^\rho(\rho+\xi)(Kw+K_1w^3)\,\mathrm{d}\rho\right.$$

$$\left.-(1+\xi)^2(\rho+\xi)Gw'\right]=\frac{(\rho+\xi)}{(1+\xi)^4}\left[\int_0^\rho\left\{\frac{12(\beta-v_\theta^2)}{\beta}Q-\ddot{w}\right\}(\rho+\xi)\,\mathrm{d}\rho\right] \quad (13)$$

where $(\ )'$ and $(\dot{\ })$ are derivatives with respect to ρ and τ, respectively. Equations (13) have to be solved subject to the boundary conditions for the immovable clamped and simply supported outer edges and the free inner edge, namely

$$\rho=0:\quad \phi=0,\quad w''+v_\theta w'/\xi=0$$

$$\rho=1:\quad w=0,\quad \phi'-v_\theta\phi/(1+\xi)=0 \quad (14)$$

$$w'=0\text{ (clamped) or } w''+v_\theta w'/(1+\xi)=0\text{ (simply supported).}$$

The dimensionless stress σ is related to the dimensional stress σ^* by

$$\sigma=\frac{\sigma^*}{E_r}\left(\frac{a}{h}\right)^2.$$

3. METHOD OF SOLUTION

The method of solution is akin to the one employed by Dumir and Khatri (1985). For static analysis, the inertia term is set to zero. The load is incremented in small steps and the governing equations (13), subject to boundary conditions (14), are solved iteratively at each step by linearizing the non-linear terms as

$$(w'\phi)_J=w'_{J_p}\phi_J,\quad (w')^2_{J_p}=w'_Jw'_{J_p}. \quad (15)$$

The predicted term w'_{J_p} is taken as the mean of its value at the two preceding iterations. For the first iteration w'_{J_p} is extrapolated quadratically from w' at the three preceding steps

$$W'_{J_p}=A_1w'_{J-1}+A_2w'_{J-2}+A_3w'_{J-3} \quad (16)$$

where A_1, A_2, A_3 are: $1, 0, 0\ (J=1);\ 2, -1, 0\ (J=2);\ 3, -3, 1\ (J\geqslant 3)$. An orthogonal point collection method has been used for spatial discretization with the zeros of a Legendre polynomial as collocation points. For N collocation points, w and ϕ are approximated at step J as

$$w=\sum_{m=1}^{N+3}\rho^{m-1}a_m,\quad \phi=\sum_{m=1}^{N+2}\rho^{m-1}b_m. \quad (17)$$

The five boundary conditions (14) and the $2N$ collocation equations for differential equations (13) constitute $2N+5$ *equations for the* a's and b's. The iterations are continued until the difference between the values of $w(0)$, $\phi'(0)$, $\phi'(1)$ at successive iterations is less than 0.1%.

The maximum response $w(0)_{max}$ under a uniformly distributed step load is obtained from the static response. It is based on the assumptions that at the instant of maximum deflection at the free edge, the annulus has zero velocity and that the deflected shape is the same as that under a static load which causes the same deflection at the free edge. If the static load Q results in a deflection $w(0)$ with the total potential energy $U(Q)$ of the plate and the foundation, then the step load Q_0 which will yield $w(0)_{max}$ equal to $w(0)$ is given by the energy balance equation

$$\int_b^a W(r)q_0 2\pi r\,\mathrm{d}r = 2\pi E_r h^5 \frac{Q_0}{a^2}\int_0^1 w(\rho)(\rho+\xi)\,\mathrm{d}\rho$$

$$= 2\pi E_r h^5 \frac{Q_0}{a^2}\left[\sum_{m=1}^{N+3}\left(\frac{1}{m+1}+\frac{\xi}{m}\right)b_m\right] = U(Q). \tag{18}$$

One static analysis is sufficient to get the maximum response to the whole set of step loads, instead of separate transient solutions for each step load.

As a check on the accuracy of the above method, transient analyses for step loads are also conducted. The inertia term in equations (13) is discretized using the Newmark-β scheme with the parameters corresponding to the average acceleration method. The time is incremented in small steps and the equations are solved iteratively at each step as in the static case.

4. RESULTS AND CONCLUSIONS

Convergence studies, not reported here for brevity, have shown that six collocation points and a time step size $\Delta\tau = 0.002$ yield accurate results. The present transient results are compared with the results of Nath and Jain (1983) in Table 1. The results are given for the boundary condition of zero shear at the free edge in the plate as well as zero generalized shear force in the plate and the foundation at the free edge. The present results agree very well with the results of Nath and Jain (1983). It is seen that for the Pasternak foundation, the deflection of the plate with the consistent boundary condition of zero V_r at the hole is much smaller than the deflection of the plate with zero shear in the plate at the hole.

Results are presented for clamped and simply supported annuli with zero V_r at the free edge, for annular ratio $b/a = 1/3$, orthotropic parameter $\beta = 1$ and 3; and several combinations of foundation parameters K, G and K_1. Poisson's ratio ν_θ is taken as 0.25. It may be noted that the effect of the inertia of the foundation is to modify the effective density of the plate (equations (7) and (12)). Thus the transient response to a step load including the inertia of the foundation remains the same function of non-dimensional time τ, but with a different non-dimensional time scale factor.

The maximum response $w(0)_{max}$ obtained from the dynamic analysis is compared in Table 2 with the approximate $w(0)_{max}$ predicted from the static analysis. It is evident that the approximate method gives acceptable results for engineering applications, the error being less than 4%. The maximum radial bending and membrane stresses at the support for clamped plates subjected to step load are given in Table 3 along with the approximate values predicted from static analysis. The results of transient analysis are in good agreement with the results of static analysis. Similar comparisons of maximum circumferential bending

Table 1. Comparison of maximum transient response $w(0)_{max}$ ($b/a = 1/3$, $\nu_\theta = 1/3$, $\beta = 1/2$, $Q_0 = 20$, $K_1 = 0$)

		$w(0)_{max}$					
		Clamped			Simply supported		
		Zero shear force in plate at hole		Zero generalized shear force V_r at hole	Zero shear force in plate at hole		Zero generalized shear force V_r at hole
$k[(a-b)^4]/D$	$g[(a-b)^4]/D$	Nath and Jain (1983)	Present	Present	Nath and Jain (1983)	Present	Present
100	0	0.2630	0.2624	0.2624	0.3104	0.3112	0.3112
100	25	0.1715	0.1711	0.1164	0.1933	0.1930	0.1416
100	50	0.1288	0.1288	0.0771	0.1427	0.1428	0.0924
150	0	0.1934	0.1937	0.1937	0.2029	0.2024	0.2024
150	50	0.1055	0.1056	0.0691	0.1137	0.1138	0.0793

Table 2. Maximum deflection under step load $Q_0(K_1 = 0, b/a = 1/3)$

K	G	$\beta = 1$ $w(0)_{max}$	Approx. $w(0)_{max}$	Percentage error	$\beta = 3$ $w(0)_{max}$	Approx. $w(0)_{max}$	Percentage error
			(a) Clamped, $Q_0 = 15$				
0	0	1.999	1.985	−0.7	1.525	1.533	0.5
5	0	1.769	1.777	0.5	1.383	1.390	0.5
10	0	1.566	1.586	1.3	1.249	1.260	0.9
5	1	1.344	1.346	0.1	1.118	1.117	−0.1
10	1	1.203	1.203	0.0	1.007	1.014	0.7
5	2	1.053	1.062	0.9	0.912	0.912	0.0
10	2	0.944	0.947	0.3	0.829	0.834	0.6
			(b) Simply supported, $Q_0 = 10$				
0	0	2.263	2.261	−0.1	1.752	1.732	−1.1
5	0	1.964	1.957	−0.4	1.539	1.522	−1.1
10	0	1.691	1.670	−1.2	1.345	1.324	−1.6
5	1	1.461	1.458	−0.2	1.206	1.192	−1.2
10	1	1.240	1.233	−0.6	1.042	1.033	−0.9
5	2	1.091	1.104	1.2	0.944	0.933	−0.6
10	2	0.940	0.936	−0.4	0.824	0.822	−0.2

Table 3. Maximum radial stress at support for clamped plates under step load $Q_0 = 15$ $(K_1 = 0, b/a = 1/3)$

β	K	G	σ^b_{max}	Approx. σ^b_{max}	Percentage error	σ^m_{max}	Approx. σ^m_{max}	Percentage error
1	0	0	15.80	15.93	0.8	1.535	1.526	−0.6
	5	0	14.03	13.89	−1.0	1.215	1.214	−0.1
	10	0	12.50	12.22	−2.2	0.954	0.964	1.0
	5	1	11.62	11.59	−0.3	0.735	0.734	−0.1
	10	1	10.55	10.46	−0.9	0.589	0.588	−0.2
	5	2	9.90	10.13	2.3	0.470	0.469	−0.2
	10	2	9.55	9.33	−2.3	0.386	0.386	0.0
3	0	0	13.85	14.01	1.2	1.639	1.641	0.1
	5	0	12.46	12.51	0.4	1.347	1.345	−0.1
	10	0	11.44	11.25	−1.7	1.103	1.103	0.0
	5	1	10.93	10.82	−1.0	0.906	0.907	0.1
	10	1	10.09	9.90	−1.9	0.752	0.752	0.0
	5	2	9.72	9.64	−0.8	0.632	0.634	0.3
	10	2	8.88	8.94	0.7	0.537	0.534	−0.6

Table 4. Maximum circumferential stress at hole for simply supported plates under step load $Q_0 = 10$ $(K_1 = 0, b/a = 1/3)$

β	K	G	σ^b_{max}	Approx. σ^b_{max}	Percentage error	σ^m_{max}	Approx. σ^m_{max}	Percentage error
1	0	0	4.041	3.956	−2.1	9.042	9.055	0.1
	5	0	3.552	3.444	−3.0	6.790	6.780	0.0
	10	0	3.233	2.932	−9.3	5.015	4.932	−1.7
	5	1	2.386	2.288	−4.1	3.813	3.804	−0.2
	10	1	2.048	1.898	−7.3	2.750	2.731	−0.7
	5	2	1.600	1.519	−5.1	2.170	2.162	−0.4
	10	2	1.321	1.270	−3.9	1.610	1.602	−0.5
3	0	0	7.923	7.536	−4.9	8.891	8.669	−2.5
	5	0	6.891	6.524	−5.3	6.853	6.645	−3.0
	10	0	6.035	5.558	−7.9	5.224	5.059	−3.2
	5	1	4.714	4.578	−2.9	4.161	4.105	−1.3
	10	1	4.089	3.859	−5.6	3.140	3.091	−1.6
	5	2	3.410	3.246	−4.8	2.580	2.550	−1.2
	10	2	2.804	2.758	−1.6	1.971	1.960	−0.6

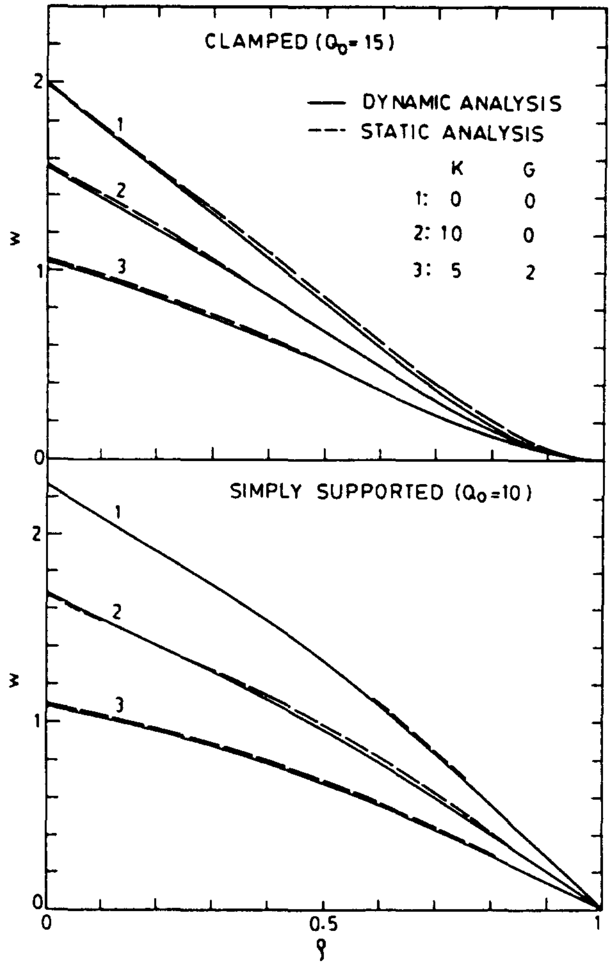

Fig. 2. Deflection profiles by static and dynamic analysis for step load (K_1 = 0, β = 1, b/a = 1/3).

and membrane stresses at the support for simply supported plates subjected to a step load are shown in Table 4. The transient results are in fair agreement with those predicted from the static analysis. The maximum error is 9.3%. The profiles of maximum deflection for clamped and simply supported isotropic plates subjected to step loads, obtained by the static and dynamic analyses are compared in Fig. 2 for three sets of foundation parameters, K and G. There is good agreement amongst the profiles obtained by static and dynamic analyses. The profiles of maximum bending and membrane stresses for plates under step loads, obtained by transient analyses are compared in Fig. 3 with the profiles obtained by static analysis. There is satisfactory agreement between the two sets of profiles.

The comparisons given in Tables 2–4 and Figs 2 and 3 have demonstrated that a fairly accurate maximum response for step loads can be obtained from the static results. This approach is used to compute the maximum deflection for a set of step loads. The static and approximate dynamic deflection response of clamped annuli with $b/a = 1/3$ is presented in Figs 4 and 5 for $\beta = 1$ and 3, respectively, for selected Winkler–Pasternak foundation stiffnesses. It is evident from these figures that the deflection decreases as β, K and G increase. The transverse plate stiffness increases with plate deflection due to the increasing action of membrane effects. Therefore, the effect of the geometric nonlinearity of the plate is to increase its transverse stiffness and thus reduce the relative effect of foundation parameters K and G in reducing deflections. Moreover, an increase in G leads to a greater reduction in the maximum deflection than an increase in K.

The static and approximate dynamic deflection response of an isotropic simply supported plate with $b/a = 1/3$ is shown in Fig. 6. It may be noted from Figs 4 and 6 as well

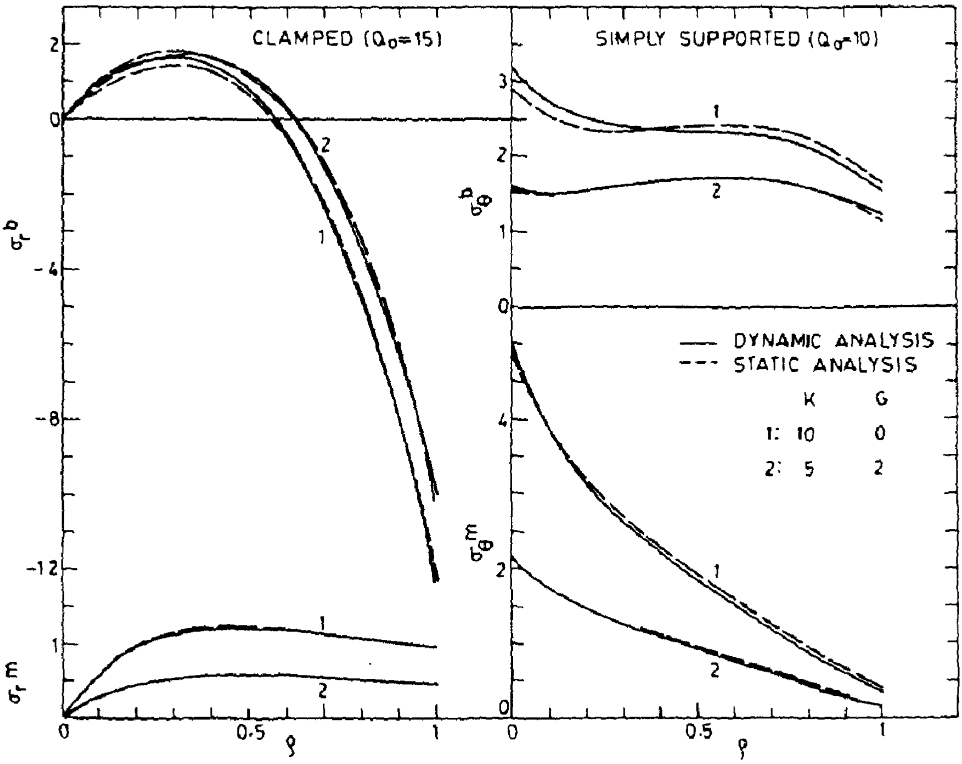

Fig. 3. Stress profiles by static and dynamic analysis for step load ($K_1 = 0$, $\beta = 1$, $h/a = 1/3$).

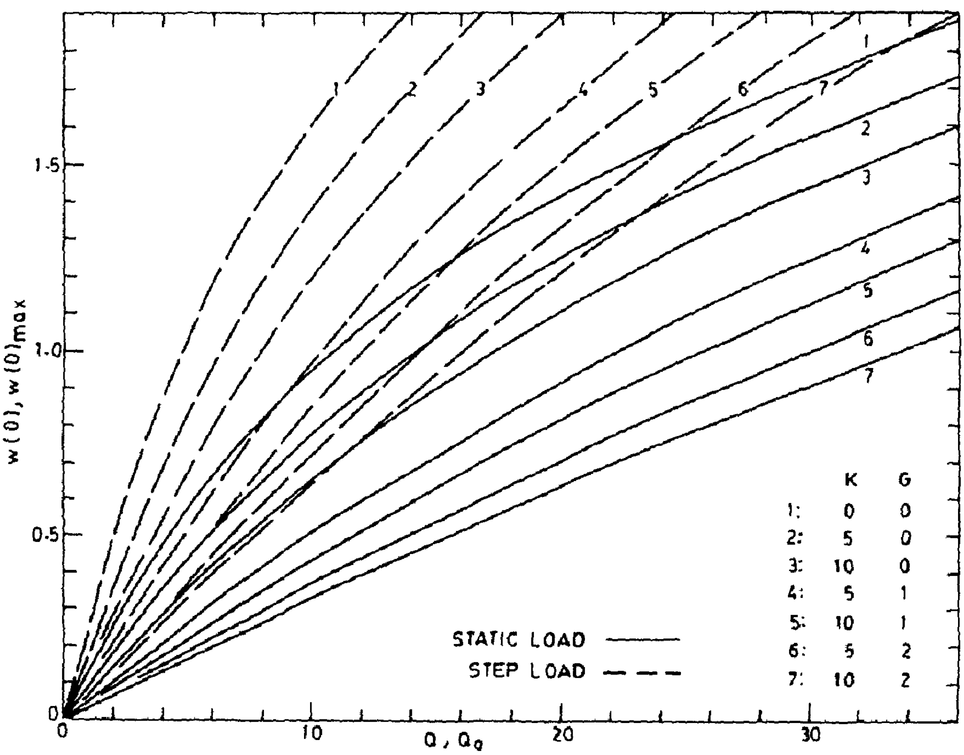

Fig. 4. Deflection response of isotropic clamped plate ($\beta = 1$, $K_1 = 0$, $h/a = 1/3$).

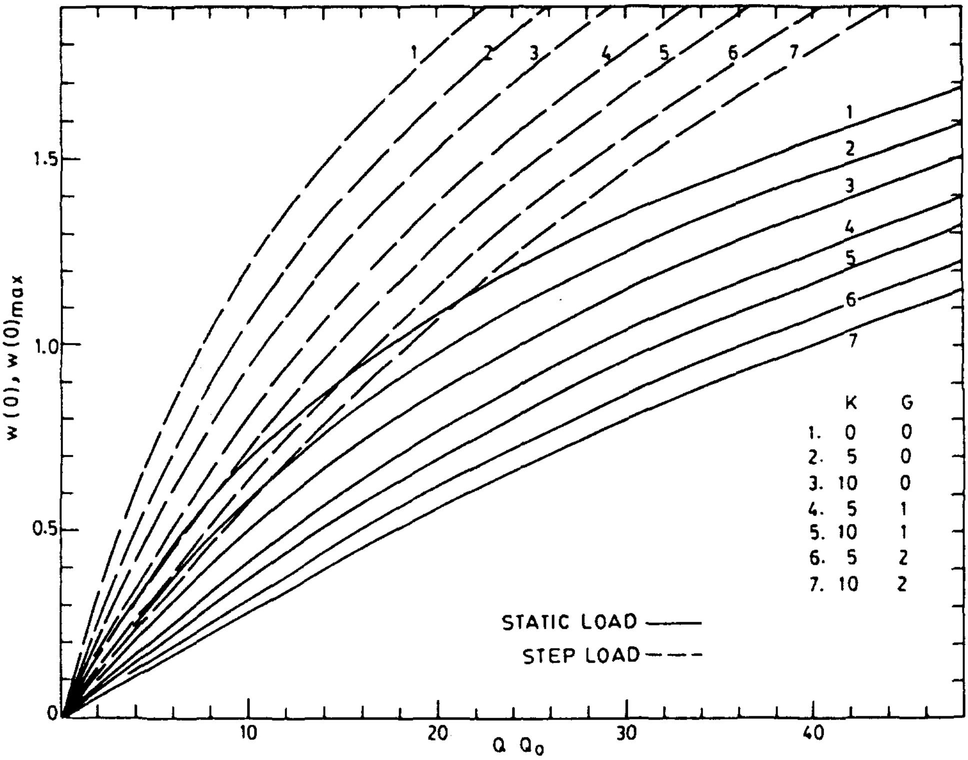

Fig. 5. Deflection response of orthotropic clamped plate ($\beta = 3$, $K_1 = 0$, $b/a = 1/3$).

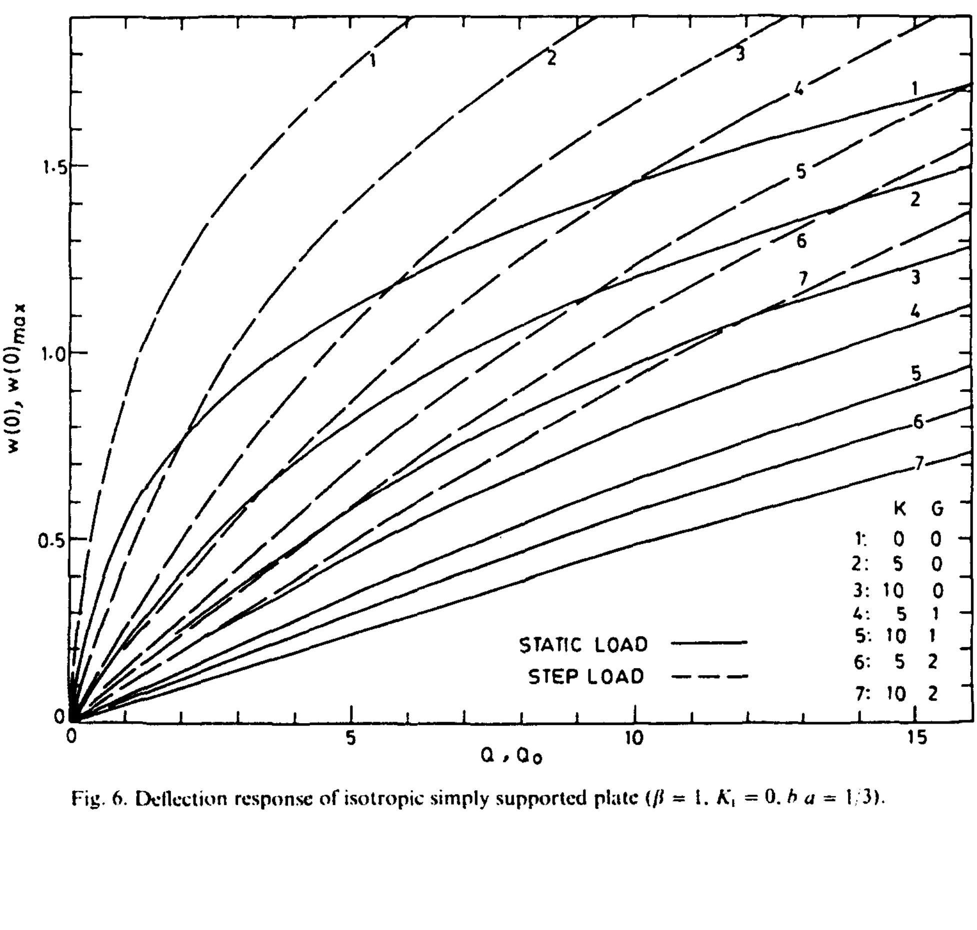

Fig. 6. Deflection response of isotropic simply supported plate ($\beta = 1$, $K_1 = 0$, $b/a = 1/3$).

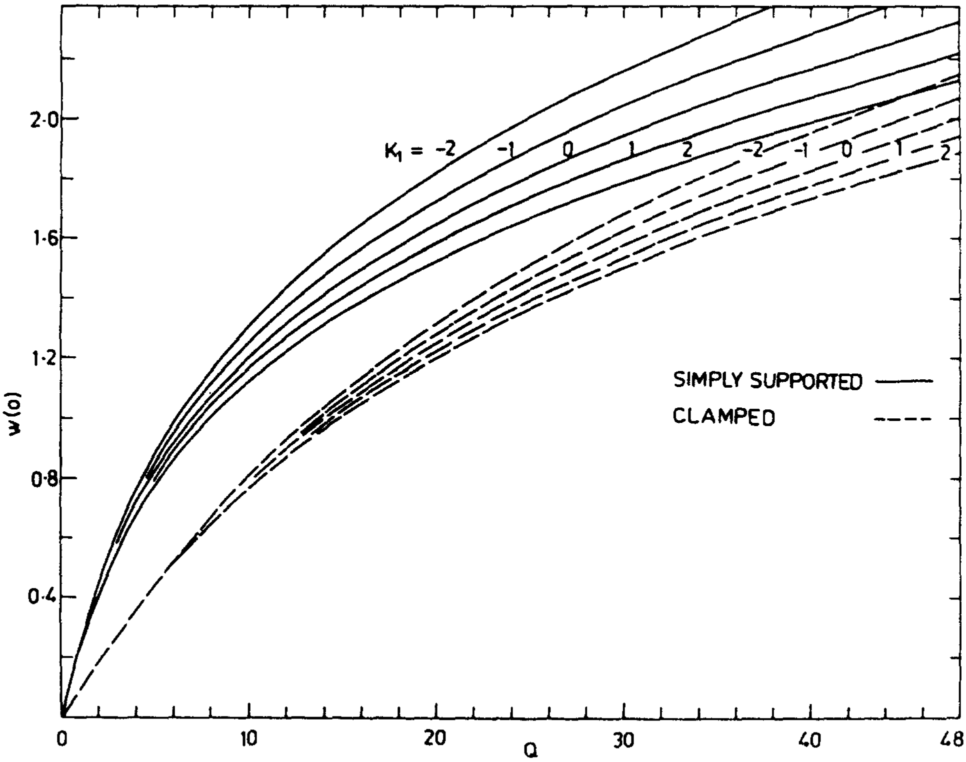

Fig. 7. Static response of plates on non-linear Winkler foundation ($b/a = 1/3$, $\beta = 1$, $K = 5$, $G = 0$).

as Table 2 that the foundation parameters K and G produce a greater reduction in the maximum deflections of simply supported plates than of clamped plates. This is due to the lower transverse stiffness of the simply supported plates. The static response of isotropic clamped and simply supported plates with $b/a = 1/3$ attached to a non-linear Winkler foundation is depicted in Fig. 7. The deflection decreases as the non-linear parameter K_1 increases. Parameter K_1 has a greater effect on the maximum deflection for the simply supported plates than for the clamped plates.

It may be concluded that sufficiently accurate maximum dynamic deflections produced by a whole set of step loads may be computed from a single geometrically non-linear static analysis for clamped and simply supported annular orthotropic plates resting on elastic foundations. This method is very economical compared with computing separate transient solutions for each step load. An increase in G leads to a greater reduction in the maximum deflection than an increase in K.

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